Transient Response of Joint-Dominated Space Structures: A New Linearization Technique

G. A. Foelsche,* J. H. Griffin,† and J. Bielak‡
Carnegie Mellon University, Pittsburgh, Pennsylvania

A new, efficient linearization method is presented for calculating the transient response of nonlinear systems due to initial disturbances. The method is an extension of the describing function approach in which the steady-state response of the system is calculated by representing the nonlinear element, typically joints in the case of space structures, by impedances that are functions of the amplitude of response. Thus, the problem of solving the differential equation for the steady-state response becomes one of solving a set of nonlinear algebraic equations involving the steady-state amplitudes and phases of the system. It is shown that for the transient case, the steady-state impedances can be averaged over the range of response in order to provide equivalent values of stiffness and damping that for a given set of initial displacements may be treated as being constant for purposes of calculating system response. Single-degree-of-freedom systems are used first to demonstrate the method and then to develop an approach for optimizing the joint's characteristics so as to minimize transient response times. The use of this method for response estimation and optimization in multiple-degree-of-freedom systems is subsequently investigated.

Nomenclature

B	= amplitude of x (slowly varying with time)
c,c_1,C	= viscous damping
c_{eff}	= effective value of instantaneous system damping
f_n	= force from nonlinear friction element
F_c,F_s	= fundamental Fourier coefficients of f_n
$k,k_i,\boldsymbol{K},\boldsymbol{K}_q$	= stiffness
k_d	= stiffness of nonlinear friction element
k_{eff}	= effective value of instantaneous system stiffness
m, $m_i,\boldsymbol{M}$	= mass
N	= normal load on friction contact
t	= time
Δt	= half-period
μ_i	= eigenvectors
$x,\boldsymbol{x},x_i$	= displacements
$\boldsymbol{y},y_i$	= modal displacements
ϵ	$= k_d/(k_d+k)$
γ_i	= eigenvalues of linearized system
μ	= coefficient of friction
ϕ	= phase lag
θ	$= \omega t - \phi$
θ^*	= θ value at which nonlinear element changes from stick to slip
ω	= frequency
ω_d,ω_n	= natural frequencies
ζ	= fraction of critical damping

Superscripts

*	= nondimensionalized quantities
$\bar{(\ )}$	= quantities averaged over a range of response

Received Sept. 14, 1987; revision received Feb. 1, 1988.

*Graduate Student, Department of Mechanical Engineering.
†Professor, Department of Mechanical Engineering.
‡Professor, Department of Civil Engineering.

Introduction

An important problem related to the design of space structures is that of predicting their dynamic response. This process is particularly difficult when dealing with structures that contain a large number of joints that exhibit nonlinear, hysteretic behavior. For example, this may be the case for prefabricated truss structures that are designed to collapse into a dense package for transportation to orbit. The truss is then expanded in space by using joints that are especially designed to rotate and lock into place. The dynamic response of such structures is said to be joint-dominated if the amount of damping or the stiffness of the system is strongly affected by the joints' behavior. If damping in the system is primarily due to joint hysteresis, then joint behavior controls the amplitude of the steady-state response as well as the rate at which transients decay. Additionally, in some cases, joint flexibility can significantly reduce the stiffness of the structure, thus reducing its natural frequencies and altering the associated mode shapes. This paper discusses a new approach that may be used to efficiently estimate the transient response of such systems.

The transient response of nonlinear systems is usually calculated by time integration methods that employ finite differences in time; see, for example, Hughes.[1] This approach has two disadvantages when it is applied to the design of joint-dominated structures. First, it is computationally intensive. This would be the case especially for the type of complex three-dimensional truss structures that are proposed for space applications, since they have a large number of degrees of freedom. Second, the problem is nonlinear, and, consequently, the solutions lack generality. For example, the rate of decay of a transient would depend on the specific magnitude and distribution of the assumed initial displacements and velocities of the structure. Since the number of degrees of freedom is large, it is not reasonable to consider all possible initial conditions. So one difficulty faced by the design engineer is in trying to select those conditions that are of critical importance to the design and to simulate the corresponding system response when there is such a large number of possible cases to consider. Neither the selection nor the simulation process is particularly feasible if time integration is the only procedure available for calculating the response of the system.

In this paper, an approximate method is developed for estimating the transient response of nonlinear systems in terms of linearized modes of response. Its advantages are that it is computationally more efficient than the time integration method and that it is possible to view the design problem in the more traditional physical terms of modal response. For example, if it is most important to damp the larger-amplitude, low-frequency response, one can easily focus on that issue by isolating the response of the first few linearized modes. Consequently, by using this approach, the design problem should become more tractable. The major drawback of the approximate method is loss of accuracy. It is our view that both approximate methods and time integration have their roles in design. Approximate methods provide efficient tools for performing parametric studies, and they supply physical insights into how to optimize system performance that are not easily inferred from strictly numerical methods. Time integration provides a method for assessing the accuracy of the approximate solution for key simulations and for fine-tuning the final design.

In the procedure presented here, the nonlinear system is approximated by an equivalent linear system in which the system parameters are constant over the range of transient response. The method is an extension of the describing function approach developed originally for calculating the steady-state harmonic response of nonlinear systems. For a detailed description of the describing function approach, see the book by Gelb and Vander Velde.[2] In this approach, the "long time" response is assumed to be essentially harmonic, and the nonlinear element is represented by impedances that depend on the amplitude of response. The component of the impedance function that is in-phase with the displacement can be interpreted as a nonlinear stiffness, whereas the component that is out-of-phase is proportional to the damping in the element. As a result, the problem of solving the original nonlinear differential equation for the steady-state response reduces to one of solving a set of nonlinear algebraic equations for the steady-state amplitudes and phases associated with the various degrees of freedom of the system. The resulting set of nonlinear algebraic equations can be solved readily for each frequency using standard numerical methods (e.g., those of Ref. 3). Alternatively, the original differential equation could be solved using step-by-step time integration techniques over a sufficiently long period of time so as to establish steady-state response. This latter course is, in general, computationally much less efficient (frequently one to two orders of magnitude) than using describing functions. The method introduced in this paper is essentially an extension of the describing function approach for estimating transient response. In this new method, the steady-state impedances are averaged over the range of response in order to provide equivalent values of stiffness and damping that for a given set of initial conditions may be treated as constants for purposes of calculating system response. We refer to this approach as the Amplitude Averaging (AA) Method because the impedances are averaged over a range of amplitudes. Once equivalent parameters are identified for the system, conventional methods can be employed for analyzing the resulting linear system.

The AA Method follows conceptually from an efficient time integration procedure presented by Sinha and Griffin[4] in which single time steps were used to step from one peak to the oscillation to the next—we refer to their method as a Half-Cycle method. It differs from the AA Method in that it uses "instantaneous" values of impedance that are updated at each time step instead of the amplitude averaging used by the AA Method. Their approach, in turn, is based on the idea that the response may be approximated as a sinusoid in which the amplitude and phase vary slowly with time (see, Caughey[5]). Related studies have been summarized by Iwan and Gates.[6]

In the next section, the AA Method is illustrated by applying it to a single-degree-of-freedom (SDF) system exhibiting the bilinear hysteretic behavior typically associated with Coulomb friction. While the method is not restricted to this type of nonlinearity, this behavior was selected for analysis because it is representative of the type of severe nonlinearity that occurs in actual joints. As a first step in the analysis, a half-cycle method similar to the method used by Sinha and Griffin is presented in order to illustrate the linearization process and develop instantaneous values of the nonlinear element's stiffness and damping. The instantaneous values of the element's parameters, which effectively characterize the joint's properties at a given amplitude of response, are then averaged over a range of amplitudes in order to calculate the constant stiffness and damping values used in the AA Method. (In the example used here to demonstrate the approach, the instantaneous joint properties are calculated analytically. Actual joint properties may be calculated from laboratory tests that measure the joint's steady-state hysteresis curves. See Crawley[7] for an example of steady-state joint characterization.) Since an equivalent, constant damping is thus determined for the system, it may be used to select joint characteristics (the friction slip load in this example) so as to maximize average joint damping and minimize transient response times. In the second part of the paper, a general approach that may be used for multiple-degree-of-freedom (MDF) systems is given and applied to the two-body problem. In each case, the accuracy of the approach is assessed by comparing results from the approximate method with those obtained using standard time integration methods.

Single-Degree-of-Freedom Systems

In order to demonstrate the AA Method, we develop a solution for the bilinear hysteretic SDF system depicted in Fig. 1. The equation of motion for the system is

$$m\ddot{x} + c\dot{x} + kx = -f_n \tag{1}$$

where f_n is the nonlinear force from the friction element and the dots represent differentiation with respect to time. During oscillation, the friction joint remains locked until the magnitude of the spring force $|k_d(x-y)|$ equals the friction force μN. The joint then slips with a constant resistive force of magnitude μN until the mass reaches an extremum of oscillation, at which point the joint locks up again. The magnitude of the relative displacement $(x-y)$ required to cause slip is designated as x_{crit}, where

$$x_{\text{crit}} = \frac{\mu N}{k_d} \tag{2}$$

Half-Cycle Method

If the transient response is approximately a sinusoid that has an amplitude and phase that vary slowly with time, then over a limited time span it may be approximated as $x = B\cos\theta$, where $\theta = \omega t - \phi$. We assume that the nonlinear force f_n exhibits the same periodicity. Expressing f_n in a Fourier series, we obtain (see Menq and Griffin[8])

$$f_n = F_c(B)\cos\theta + F_s(B)\sin\theta + (\text{higher harmonics}) \tag{3}$$

where, for the case of Coulomb friction considered,

$$F_c(B) = (k_d B/\pi)[\theta^*(B) - \tfrac{1}{2}\sin 2\theta^*(B)] \tag{4}$$

$$F_s(B) = -(4\mu N/\pi)[1 - (\mu N/k_d B)] \tag{5}$$

$$\theta^*(B) = \cos^{-1}[1 - (2\mu N/k_d B)], \qquad 0 \le \theta^* \le \pi \tag{6}$$

In the case of experimental joint data, the Fourier coefficients can be calculated numerically from hysteresis curves using numerical integration.

If we keep only the fundamental harmonics and truncate Eq. (3) after the first terms, we can express f_n as

$$f_n = (F_c(B)/B)x + (-F_s(B)/\omega B)\dot{x} \tag{7}$$

Thus, Eq. (1) becomes

$$m\ddot{x} + [c - (F_s/\omega B)]\dot{x} + [k + (F_c/B)]x = 0 \tag{8}$$

or

$$m\ddot{x} + c_{\text{eff}}(B)\dot{x} + k_{\text{eff}}(B)x = 0 \tag{9}$$

where c_{eff} and k_{eff} are defined in a manner consistent with Eq. (8) and, because they pertain to a specific amplitude, are referred to as the instantaneous damping and stiffness of the bilinear spring. Alternatively, we may write Eq. (9) as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{10}$$

where ζ and ω_n are both functions of B:

$$\zeta(B) = c_{\text{eff}}(B)/2[m\, k_{\text{eff}}(B)]^{\frac{1}{2}} \tag{11}$$

$$\omega_n(B) = (k_{\text{eff}}(B)/m)^{\frac{1}{2}} \tag{12}$$

The damped natural frequency of this system may be similarly defined as

$$\omega_d(B) = \{[1 - \zeta^2(B)]k_{\text{eff}}(B)/m\}^{\frac{1}{2}} \tag{13}$$

It is observed that during transient oscillation of the nonlinear system, the response is similar to the decaying sinusoid seen in linear analysis. It is reasonable to assume that the motion of the nonlinear system from one extremum of oscillation to the next extremum is representable as the decay of a linear system over a half-cycle. As an example, we consider a system decaying from initial conditions of some initial displacement B_o and zero initial velocity. Let B_i denote the amplitude of the ith extremum (occurring at time t_i) and B_{i+1} denote the amplitude of the next extremum (at time t_{i+1}). Then, from the linear theory, the time elapsed between one extremum and the next is approximately

$$\Delta t = t_{i+1} - t_i = \pi/\omega_d(B_i) \tag{14}$$

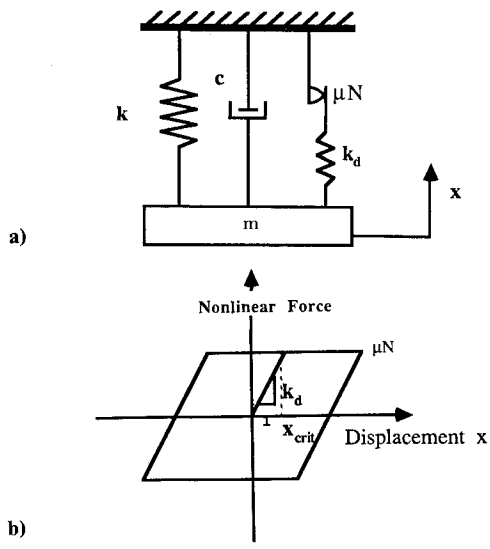


Fig. 1 Single-degree-of-freedom system.

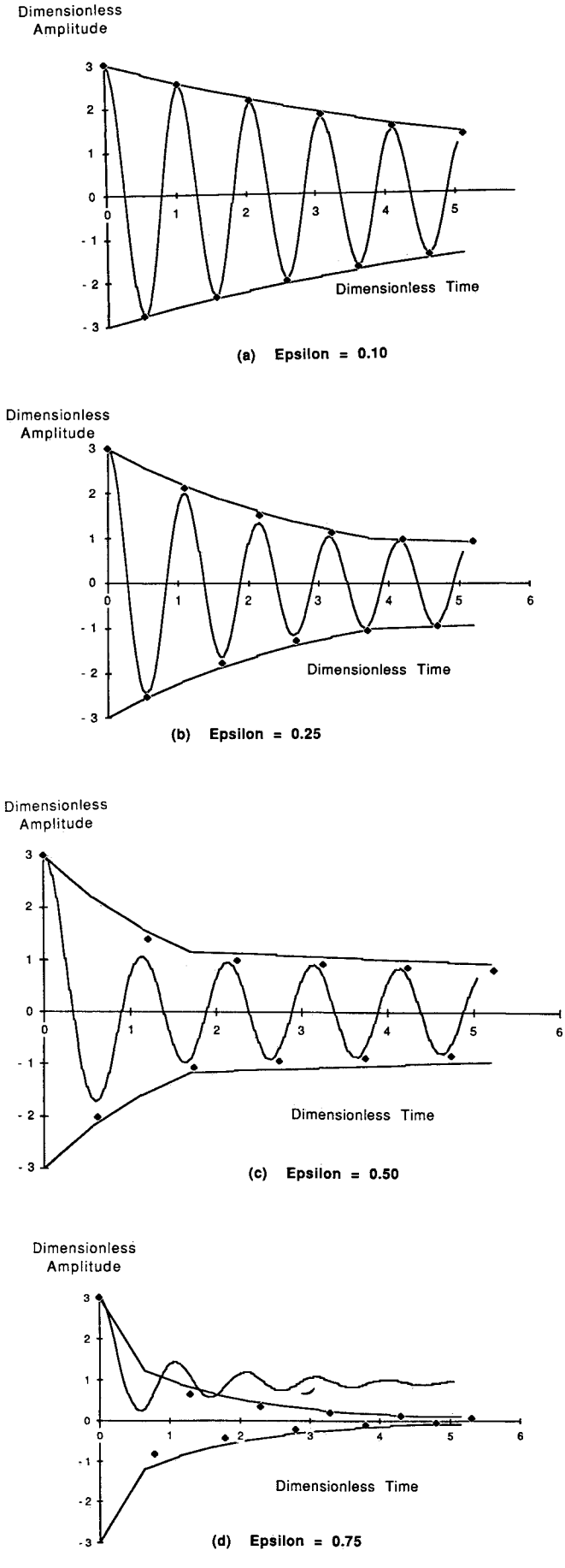


Fig. 2 Comparison of results for $m = k + k_d = \mu N/k_d = 1$ and $c = 0.02$.

and the relationship between successive peaks is

$$B_{i+1} = -B_i e^{-\zeta_i \pi/(1-\zeta_i^2)^{\frac{1}{2}}} \tag{15}$$

Given the initial amplitude B_o, we can efficiently estimate successive extrema and the time increments at which they occur from Eqs. (14) and (15). Since the extrema occur each half-cycle, we refer to this method as the Half-Cycle Method.

This Half-Cycle Method is an efficient way of approximating a numerical time integration to find extrema of transient oscillation. Numerical simulations of SDF systems show that this method is accurate for nonlinear systems in which $\epsilon = k_d/(k + k_d) \leq 0.5$ and provides a reasonable approximation for the amplitude of response for $\epsilon > 0.5$. The Half-Cycle Method results were compared to more accurate solutions generated by fourth-order Runge-Kutta time integration. The comparison of the Half-Cycle estimates with numerically generated "exact" solutions is shown in Figs. 2a–d, where the Half-Cycle estimates of the extrema of response are shown as points (the exponential decay envelopes pictured are a result of the new Amplitude Averaging Method and will be discussed later). For nonlinearities of $\epsilon > 0.5$, the system experiences an offset that is not accounted for in the Half-Cycle Method. However, the peak-to-peak amplitude estimates are approximated reasonably well despite this offset. It may be observed that for these simulations, this approach yields conservative estimates of system behavior in that it overestimates the amplitudes of response.

Amplitude Averaging Method

We now introduce a new linearization called the AA Method. In this approach, the half-cycle values of damping and stiffness are averaged over the entire response range of interest. Consequently, while these average values are nonlinear functions of the initial displacement, they are constants as far as the transient response analysis is concerned.

The AA Method is also based on Eqs. (3–6). Linear, constant parameters are derived by averaging c_{eff} and k_{eff} from Eq. (9) over the nonlinear range of oscillation, i.e.,

$$\bar{c}_{\text{eff}}(B_o) = c + \bar{c}_e(B_o) = c + \int_{x_{\text{crit}}}^{B_o} \frac{-F_s(B)}{B\omega} dB/(B_o - x_{\text{crit}}) \tag{16}$$

$$\bar{k}_{\text{eff}}(B_o) = k + \bar{k}_e(B_o) = k + \int_{x_{\text{crit}}}^{B_o} \frac{F_c(B)}{B} dB/(B_o - x_{\text{crit}}) \tag{17}$$

where ω in Eq. (16) is given by Eq. (12). After averaging, Eq. (8) becomes

$$m\ddot{x} + [c + \bar{c}_e(B_o)]\dot{x} + [k + \bar{k}_e(B_o)]x = 0 \tag{18}$$

For the example of Coulomb friction, the analysis can be simplified by expressing the averaged properties in terms of several nondimensional parameters (normalized quantities denoted by * superscript):

$$\bar{k}_e(B_o) = k_d \bar{k}_e^*(B_o^*) \tag{19}$$

$$\bar{c}_e(B_o) = [c_o/\bar{\omega}^*(B_o^*)]\bar{c}_e^*(B_o^*) \tag{20}$$

where

$$c_o = 2m\omega_o = 2(mk_d)^{\frac{1}{2}} \tag{21}$$

$$\bar{\omega}^*(B_o^*) = [(k/k_d) + \bar{k}_e^*(B_o^*)]^{\frac{1}{2}} \tag{22}$$

and

$$B_o^* = B_o/x_{\text{crit}} = B_o k_d/\mu N \tag{23}$$

where $\bar{k}_e^*(B_o^*)$ and $\bar{c}_e^*(B_o^*)$ are the nondimensional averaged stiffness and damping of the friction element. These quantities are of particular interest because their values may be calculated in terms of the single nondimensional parameter B_o^*, the initial displacement divided by the displacement required for slip to occur.

For Coulomb friction, from Eqs. (4), (17), and (19), $\bar{k}_e^*$ is given by

$$\bar{k}_e^*(B_o^*) = \frac{1}{(B_o^* - 1)} \int_1^{B_o^*} k_e^*(B^*)\, dB^* \tag{24}$$

where

$$k_e^*(B^*) = \frac{1}{\pi}\{\theta(B^*) - \tfrac{1}{2}\sin[2\theta(B^*)]\}$$

$$\cos\theta(B^*) = 1 - (2/B^*), \qquad (B^* \geq 1)$$

and needs to be calculated only once. The nondimensional frequency $\bar{\omega}^*$ is then obtained from Eqs. (12), (17), and (19) as

$$\bar{\omega}^*(B_o^*) = [(k/k_d) + \bar{k}_e^*(B_o^*)]^{\frac{1}{2}} \tag{25}$$

Lastly, from Eqs. (5), (16), and (20), the average nondimensional damping is approximated as

$$\bar{c}_e^*(B_o^*) = \frac{1}{(B_o^* - 1)} \int_1^{B_o^*} c_e^*(B^*)\, dB^* \tag{26}$$

where

$$c_e^*(B^*) = 2(1 - 1/B^*)/(\pi B^*)$$

which, upon integration, yields

$$\bar{c}_e^*(B_o^*) = [2/\pi(B_o^* - 1)][(2/B_o^*) + \ell n(B_o^*) - 1] \tag{27}$$

Note that the parameters are averaged only over the range in which they exhibit nonlinear behavior. The system is linear for B_o^* less than one, and the response can be calculated using standard methods in that regime.

The AA Method gives a linear estimate of the nonlinear behavior of the system. Results from the AA linearized systems are shown in Figs. 2a–d as exponential decay envelopes and are plotted with the Half-Cycle Method and Runge-Kutta results for comparison. As with the Half-Cycle Method, we again see conservative estimates of system amplitudes, and again the estimates are more precise for weakly nonlinear systems.

An important result of the AA Method is shown in Fig. 3. This figure shows the normalized quantities $k_e^*(B_o^*)$, $c_e^*(B_o^*)$, $\bar{k}_e^*(B_o^*)$ and $\bar{c}_e^*(B_o^*)$ in terms of the nondimensional initial displacement B_o^*. It is observed that there are optimal system configurations that maximize either instantaneous or average damping in the system. These optimality conditions depend

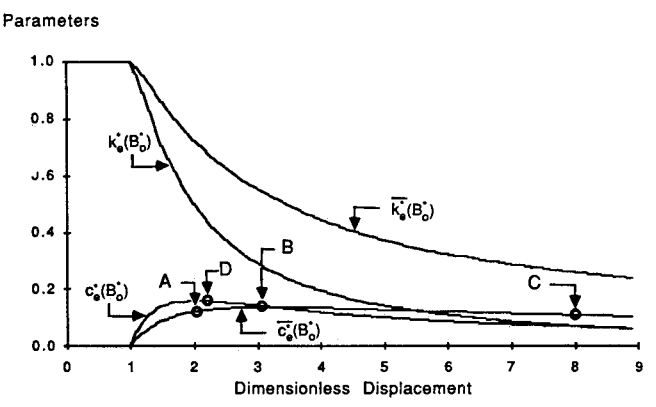


Fig. 3 Instantaneous and average values of stiffness and damping.

only on the nondimensional initial displacement (B_o^*). Recall from Eq. (23) that B_o^* is dependent on the system parameters. Thus, by adjusting just one of these parameters, e.g., normal load, it is possible to optimally damp an existing system. For example, the points A, B, and C on the average damping curve can be thought of as three systems that are identical except for the tightness of the friction joint, as indicated by the normal load (normal load $= N, N_A > N_B > N_C$). Comparing the linear average damping terms provided by the AA Method for the three systems, it is seen that system B, with normal load N_B, has the highest value of average damping and is an optimally damped system for this set of basic parameters. The optimality of system B in an average sense was confirmed by fourth-order Runge-Kutta simulations. The results for the three systems A, B, and C are shown in Fig. 4, where successive extrema have been connected to form an envelope of decay. The normal load N_B is not simply the largest or smallest normal load that could be applied, nor is it equal to the load that would be required to optimally damp only the first oscillatory swing in the transient motion (this normal load would correspond to the conditions of system D of Fig. 3, which maximize instantaneous damping).

Runge-Kutta simulations confirm the optimality results that were readily provided by the AA Method. We observe, however, that the optimality result would not have been nearly so obvious if an exact time integration alone had been used to investigate this transient behavior. It is also noted that the AA Method provides general results in that it allows for parameter-based comparisons of different systems. Numerical time integration methods lack this generality, yielding instead results that are case-specific and thus more difficult to interpret when comparing systems.

The AA Method leads to systems that are optimized, in an averaged sense, over the entire range of nonlinear behavior. The Half-Cycle Method may be used to generate systems that are optimal in a "first swing" sense. It is also possible to optimize the system over other select ranges of nonlinear behavior. This is done by averaging the c_{eff} and k_{eff} equations over the particular range of concern, resulting in a new curve for $\bar{c}_e^*(B_o^*)$. The optimal normal load (or other parameter) is the one that adjusts the nondimensional initial condition so as to maximize the value of this new $\bar{c}_e^*(B_o^*)$. The linearizations described so far make it possible to optimize the nonlinear system, in an average sense, over any range of nonlinear behavior.

The Amplitude Averaging Method has yielded a general result that was not obvious from numerical time integrations. In the next section, the application of the AA Method to multiple mass systems is developed, and it is shown that the principles of generality-of-results and optimal damping still apply, only that they now apply in a modal sense.

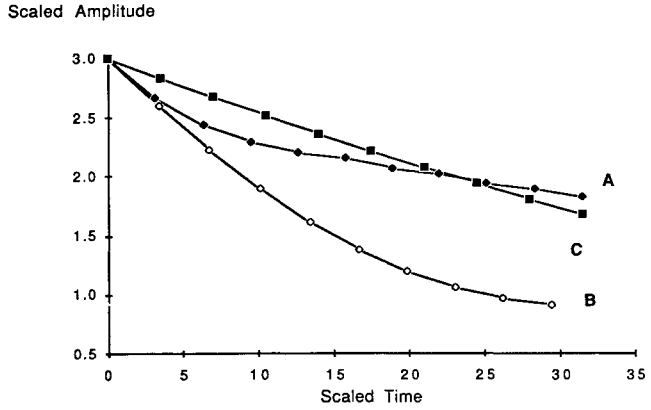


Fig. 4 Response envelopes indicate optimum damping.

Multiple-Degree-of-Freedom Systems

The transient analysis of MDF nonlinear systems using the AA Method can be accomplished by representing the system in modal form in terms of a sum of SDF nonlinear systems. The SDF components are linearized separately and are then combined to form a linear representation of the MDF system.

In the linearization process, a nonlinear friction damper will be replaced by linear elements that approximate its behavior, as depicted in Fig. 5. The original 2DF system incorporates a friction damper as a nonlinear element and will serve to illustrate the application of the AA Method to MDF nonlinear systems. The equations of motion for the system may be written in matrix form as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -f_n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}$$

MDF Linearization

In the SDF analysis, we were able to regard the initial amplitude across the friction damper as the maximum distance that the joint would be stretched during transient oscillation. In the MDF system, the initial amplitude across the joint is no longer guaranteed to be the maximum span that the joint experiences. Consequently, the nonlinear system cannot be linearized by simply considering the isolated friction damper and how much it is initially displaced. Instead, one must use a modal approach in the linearization process.

It is necessary to decouple the nonlinear MDF system of Fig. 5 into two SDF nonlinear systems so that the AA Method may be applied to each of the decoupled systems separately. Decoupling of linear systems is done routinely (see, e.g., Thomson[9]), while nonlinear systems are not generally amenable to such analysis. The nonlinear decoupling and modal linearizations may be accomplished with the iterative approach summarized in the Appendix. In this approach, a set of converged eigenvalues and eigenvectors is obtained and used to form a modal (decoupled) representation of the

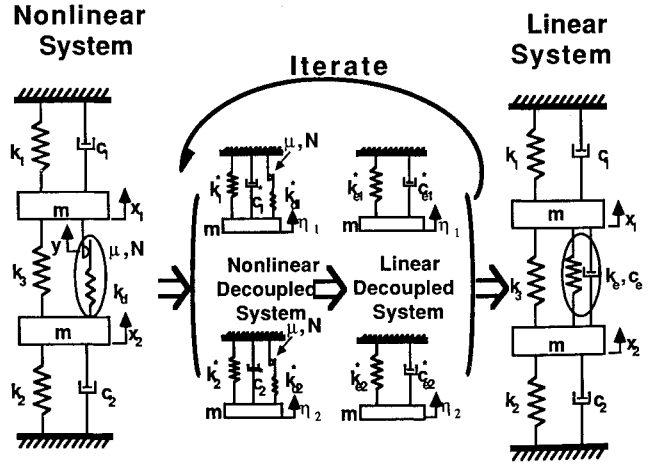


Fig. 5 Schematic of linearization process.

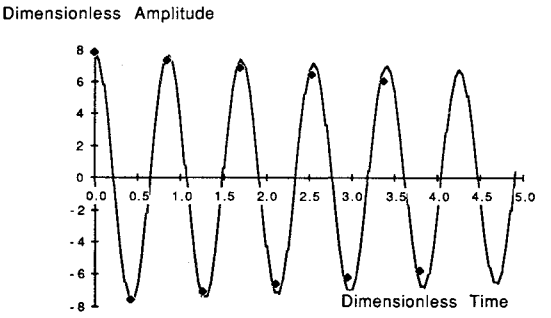
nonlinear system. The decoupled systems are linearized using the AA Method and are transformed back to the original coordinates to yield a linear MDF system.

Several approximations are made during the decoupling process. Viscous damping is generally small and is, therefore, neglected in order to avoid the inconvenience of dealing with complex eigenvectors. The complex formulation could be pursued in order to increase the overall accuracy of the linearization and estimation scheme. Another assumption made in the analysis is that each modal friction joint operates independently. This approximation significantly simplifies the calculations but introduces an additional source of error in those cases where modes interact. This latter assumption is believed to be responsible for the fact that the modal AA response estimates are no longer conservative in some instances.

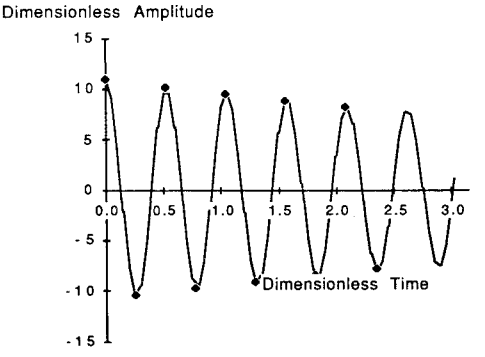
However, in general, the results obtained from the AA Method compare reasonably well with results from direct time integration. The modal comparisons for two cases are shown in Figs. 6 and 7, where the time integration solutions have been transformed from the original x coordinates into the converged modal coordinates y_1 and y_2 by using the converged eigenvectors. Overall, the exact modal solutions (represented by the continuous curves) are seen to be similar in form to decaying sinusoids centered about a zero equilibrium state. The neatness (symmetry and sinusoidal appearance) of the exact results indicate that the converged linear decoupling does in fact represent the nonlinear behavior fairly well, as the response in x coordinates would transform poorly if the modal representation were not a reasonable estimate of system response.

Several characteristics of the AA Method as applied to MDF systems are illustrated in Figs. 6, 7, and 8. First, the AA estimates are not conservative in the lowest mode when both modes are actively slipping. (It was found that if only one mode was actively slipping, the accuracy of the AA method for MDF systems was comparable to that found for SDF systems.) Second, it was found that the larger the viscous damping and nonlinearities in a mode, the less "neat" the modal response looks (this is apparent in Figs. 7b and 8b). Third, a MDF system can be modally optimized in a fashion similar to that used to optimize the SDF system. As in the SDF case, the correct manipulation of the normal load shifts the nondimensional modal initial condition to a condition that generates the optimal (modal) damping. It is noted that it is not generally possible to optimally damp both modes simultaneously, since the optimization of one mode results in a detrimental or nonoptimizing shift of the other mode. Figure 6 shows a system that is not optimally damped in either mode. Figure 7 shows the same system after the normal load has been adjusted to optimally damp the higher frequency mode (mode 2). Notice that the results are plotted on different scales in Figs. 6 and 7. In Fig. 8b, the extrema in the numerical time integration solutions for the nonoptimized mode 2 and optimized mode 2 systems have been scaled for direct comparison (extrema plotted to form decay envelope). From Fig. 8b, it is obvious that the optimization has a significant effect on modal response. The same system was optimized in mode 1 and the results depicted in Fig. 8a. Note that in this particular system the friction damping in the first mode was quite small compared to viscous damping, and, consequently, optimizing joint damping had little impact on first-mode response (an expanded scale was used to show the optimization result more clearly). The AA optimization results for both modes 1 and 2 were confirmed by Runge-Kutta simulations. Thus, although the modal AA extrema may be nonconservative in some cases, the AA indication of modal optimality remains accurate in the cases considered.

The AA Method may thus be used to optimally damp select modes of particular concern. This is helpful in the design process in that it gives an easy indication of which systems are optimal in a given situation. In practice, it may not be

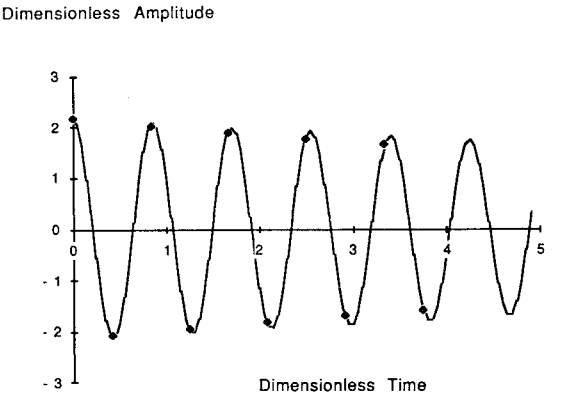


(a) Mode 1

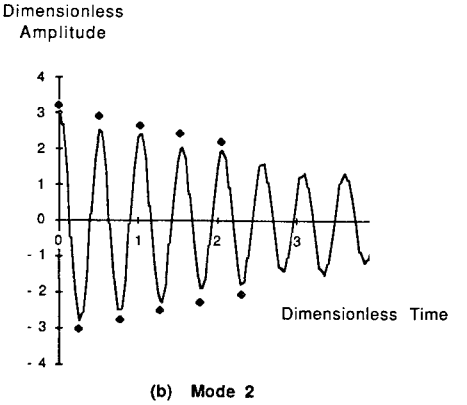


(b) Mode 2

Fig. 6 Modal response of MDF system: $k_1 = k_3 = m = 1$; $k_2 = 2$; $c_1 = c_2 = 0.01$; $k_d = 0.25$; and $\mu N = 0.1$

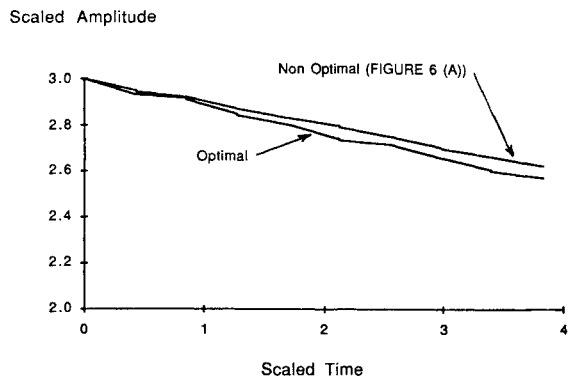


(a) Mode 1

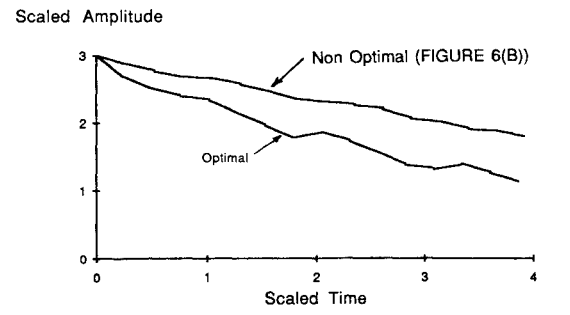


(b) Mode 2

Fig. 7 Modal response when second mode optimally damped: μN changed to 0.35.



(a) Mode 1



(b) Mode 2

Fig. 8 Comparison of modal response envelopes.

possible to obtain joints with characteristics that exactly correspond to this modeled optimality, in which case the issue may become one of selecting the most nearly optimal joint configuration from a variety of available designs.

Conclusion

This paper has discussed the specific application of the AA Method to friction-damped systems. However, the AA Method is a general linearization method applicable not only to friction-damped systems but also to other systems that exhibit nonlinear hysteretic behavior.

The AA Method is efficient and can be used to easily establish optimization conditions, subject to time integration verification. Familiar modal analysis may be applied to MDF nonlinear systems, and systems may be optimized over specific ranges of nonlinear oscillation. Furthermore, the AA Method can be used to modally optimize MDF systems in order to suppress system response over specific frequency ranges.

The AA Method may also be used as a comparison tool in the system design process. The physical parameters of the system may not be adjustable to the indicated optimal values. For example, in the case of jointed structures and friction damping, it is not generally possible to select physical joints with adjustable (optimizable) normal loads. In this situation, the task may be one of selecting joints from a variety of designs. The AA Method yields the relative averaged damping in these designs for amplitudes of response that are representative of those encountered in practice and thus may be employed as a method of comparison in order to help choose the most nearly optimal design.

The AA Method is an efficient design tool for two reasons. One is that the method is computationally efficient. In the cases considered in this paper, AA solutions could be calculated an order of magnitude more quickly than numerical time integration solutions. In addition, the method provides a system representation in terms of linearized modes, and, consequently, it becomes relatively easy to establish optimum system response. Again, it is our view that numerical time integration and the AA Method are complimentary approaches and that both have their places in the design of nonlinear systems. The AA Method is a computationally efficient approach that supplies analytical insight at the expense of accuracy, whereas the computationally intensive time integration approach provides verification and fine-tuning of the results for select cases of interest.

Appendix: Iterative Method for Decoupling and Linearizing a Nonlinear MDF System

The goal here is to represent the friction joint by equivalent linear elements for a given set of initial displacements. To this end, we must find a modal representation of the friction element that depends (as in the case of the SDF system) on the initial modal displacements. It is not possible, however, to calculate the mode shapes or eigenvectors of the system and find the initial modal displacements unless we know all the stiffness of the system (including those of the friction joint). This Appendix summarizes an iterative procedure for simultaneously establishing the eigenvectors and initial modal displacements of the system and the equivalent joint properties of the friction element.

This method neglects damping in determining the eigenvectors of the system. This is a reasonable approximation for lightly damped systems. There are seven steps in the process, which are illustrated by applying them to the two-degrees-of-freedom (2DF) system of Fig. 5.

1) Write the governing equation of the system in matrix form, neglecting viscous damping, e.g., for the 2DF system of Fig. 5:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}_o\boldsymbol{x} = f_n\begin{bmatrix}-1\\1\end{bmatrix} \tag{A1}$$

where $\boldsymbol{K}_o = \boldsymbol{K}$.

2) Find the eigenvalues and eigenvectors of the linear part of the system, momentarily disregarding the nonlinear term f_n. Form the matrix of eigenvectors $\boldsymbol{U}$, e.g.,

$$\boldsymbol{K}_o\boldsymbol{u}_i = \lambda_i\boldsymbol{M}\boldsymbol{u}_i \quad \text{eigenvalues } \lambda_1,\lambda_2$$

$$\text{eigenvectors } \boldsymbol{u}_1 = \begin{bmatrix}u_{11}\\u_{21}\end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix}u_{12}\\u_{22}\end{bmatrix} \tag{A2}$$

$$\boldsymbol{U} = [\boldsymbol{u}_1,\boldsymbol{u}_2] \ldots \text{matrix of eigenvectors}$$

3) Assume $\boldsymbol{x}$ can be represented in terms of modal coordinates $\boldsymbol{y} = (y_1,y_2)^T$, i.e.,

$$\boldsymbol{x} = \boldsymbol{U}\boldsymbol{y} \tag{A3}$$

Then

$$\boldsymbol{y} = \boldsymbol{U}^{-1}\boldsymbol{x}$$

The modal initial conditions can be found using Eq. (A3) and the orthogonality of the eigenvectors (with respect to $\boldsymbol{M}$):

$$\boldsymbol{y}_o = \boldsymbol{U}^{-1}\boldsymbol{x}_o$$

4) Using $\boldsymbol{x} = \boldsymbol{U}\boldsymbol{y}$ in Eq. (A1) yields

$$\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{y}} + \boldsymbol{K}_o\boldsymbol{U}\boldsymbol{y} = f_n\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{y}} + \boldsymbol{U}^T\boldsymbol{K}_o\boldsymbol{U}\boldsymbol{y} = \boldsymbol{U}^T f_n\begin{bmatrix}-1\\1\end{bmatrix}$$

5) Find linearized stiffness for each independent mode using the AA Method. This step yields

$$\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{y}} + \boldsymbol{U}^T\boldsymbol{K}_o\boldsymbol{U}\boldsymbol{y} = -\boldsymbol{K}_q\boldsymbol{y} - \boldsymbol{C}_q\dot{\boldsymbol{y}}$$

analogous to

$$-f_n = -(F_c/B)x + (F_s/\omega B)\dot{x} \Rightarrow -k_e x - c_e \dot{x}$$

as done in the SDF case in Eqs. (7), (8), and (16–18) in the main body of the paper. A diagonal matrix is $\boldsymbol{K}_q$, and each diagonal element is the linearized stiffness of the friction element in its respective mode; e.g., in the example 2DF system, slip occurs when

$$\mu N = k_d(x_1 - x_2) = k_{d_1} y_1 + k_{d_2} y_2$$

where

$$k_{d_i} = k_d \boldsymbol{\mu}_i [1, -1]$$

Assume that slip occurs independently in each mode. Then the modal displacement required for slip is

$$y_{i_{\text{crit}}} = \frac{\mu N}{k_{d_i}} \text{ (analogous to } x_{\text{crit}} = \mu N/k_d)$$

The modal initial conditions were found in step 3. Given the initial conditions and the slip conditions, the AA Method may be applied to each mode separately in order to linearize the decoupled systems. From the first mode, the linearized stiffness term is $-k_{e_1}$. Mode 2 yields $-k_{e_2}$.

Form $\boldsymbol{K}_q$:

$$\boldsymbol{K}_q = \begin{bmatrix} k_{e_1} & 0 \\ 0 & k_{e_2} \end{bmatrix}$$

The linearized modal stiffnesses have been found ($\boldsymbol{C}_q$ is found simultaneously but not used at this point in the analysis).

6) Convert back to original coordinates:

$$\boldsymbol{x} = \boldsymbol{U}^{-1}\boldsymbol{y}$$

Substituting into the linearized governing equation from step 5,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}_o \boldsymbol{x} = -\boldsymbol{U}^{T^{-1}} \boldsymbol{K}_q \boldsymbol{U}^{-1} \boldsymbol{x} \text{ (neglect } \boldsymbol{C}_q)$$

define

$$\boldsymbol{K}_{\text{mod}} \cong \boldsymbol{U}^{T^{-1}} \boldsymbol{K}_q \boldsymbol{U}^{-1}$$

Then

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + (\boldsymbol{K}_o + \boldsymbol{K}_{\text{mod}})\boldsymbol{x} = 0$$

7) $\boldsymbol{K}_1 = \boldsymbol{K}_o + \boldsymbol{K}_{\text{mod}}$; then

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}_1 \boldsymbol{x} = 0$$

But this system has different eigenvalues and eigenvectors than the original system considered in step 2. The linearization of step 5 was dependent on the eigenvectors of the system, which have changed after linearization. To obtain a more accurate linearization, repeat steps 1–7 using the eigenvectors from the $\boldsymbol{M}_1, \boldsymbol{K}_1$ system.

The process (steps 1–7) may be repeated as many times as needed to obtain a converged set of eigenvectors using $\boldsymbol{M}$, $\boldsymbol{K}_{(i)}$. In practice, this method usually converged in four iterations and never required more than six iterations.

While the linearized damping can be readily calculated at any iteration, it is neglected along with any original viscous damping in estimating the eigenvectors of the system. For this reason, the method is less accurate when high viscous damping or large nonlinearities are present in the system. A complex eigenvalue and eigenvector approach could be used to improve the accuracy when the damping terms are large.

In the current approach, the equivalent linear damping from the friction joint is calculated for each mode only after the eigenvectors have fully converged. This is added to the modal viscous damping to establish the total effective damping [analogous to Eq. (16)] for each mode. The response of each mode is then calculated in the same manner as for a linear single-degree-of-freedom system to obtain the comparisons indicated in Figs. 6 and 7. The physical displacements may be readily calculated from Eq. (A3).

Acknowledgments

This work was supported by NASA Langley Research Center, Grant NAG-1-612-NA64, under the direction of Mr. Lucas Horta.

References

[1]Hughes, T. J. R., *Finite Element Method,* Prentice-Hall, Englewood Cliffs, NJ, 1987, Chap. 9.

[2]Gelb, A. and Vander Velde, W. E., *Multiple-Input Describing Functions and Nonlinear System Design*, McGraw-Hill, New York, 1968.

[3]Press, W. P., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., *Numerical Recipes*, Cambridge Univ. Press, Cambridge, England, U.K., 1986, Chapter 9.

[4]Sinha, A. and Griffin, J. H., "Effects of Static Friction on the Forced Response of Frictionally Damped Turbine Blades," *Journal of Engineering for Gas Turbines and Power*, Vol. 106, Jan. 1984, pp. 65–69.

[5]Caughey, T. K., "Sinusoidal Excitation of a System With Bilinear Hysteresis," *Journal of Applied Mechanics*, Vol. 27, No. 4, Dec. 1960, pp. 640–643.

[6]Iwan, W. D. and Gates, N. C., "Estimating Earthquake Response of Simple Hysteretic Structures," *Journal of the Engineering Mechanics Division*, American Society of Civil Engineers, Vol. 105, No. EM3, June 1979, pp. 391–405.

[7]Crawley, E. F., "Identification of Nonlinear Structural Elements by Force-State Mapping," *AIAA Journal*, Vol. 24, Jan. 1986, pp. 155–162.

[8]Menq, C.-H. and Griffin, J. H. "A Comparison of Transient and Steady State Finite Element Analyses of the Forced Response of a Frictionally Damped Beam," *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 107, Jan. 1985, pp. 204–210.

[9]Thomson, W. T., *Theory of Vibrations With Applications*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1981, pp. 132–201.